On a certain Class of Groups of Transformation in Space of Three Dimensions.

By H. F. Blichfeldt.

In his "Untersuchungen über die Grundlagen der Geometrie,"* Professor Lie has investigated a remarkable class of groups of transformations in three variables, defined by the following properties: two points have one, and only one, invariant; $s > 2$ points have no invariants independent of such two-point invariants.

This class of groups belongs to a wider class in n variables defined by the following properties: *Not less than $m > 1$ points may possess invariants, while s points, $s > m$, may have no invariants independent of the m-point invariants.*

This wider class will include the Group of Euclidean Motions in space of two or three dimensions, the Group of Translations in space of n dimensions, the Group of Euclidean Motions and Similar Transformations in space of three dimensions, etc.

There is a law for such groups which can be stated as follows:

If a finite group in n variables has no invariants for less than m points $x_1 y_1 \ldots .$, $x_2 y_2 \ldots .$, $\ldots .$, $x_m y_m \ldots .$, and no invariants for $s > m$ points independent of the m-point invariants, then must the set of independent invariants in the m points $x_1 y_1 \ldots .$, $x_2 y_2 \ldots .$, $\ldots .$, $x_m y_m \ldots .$ contain all the coordinates of all these m points.

For, if one of the coordinates, as x_1, were absent, it is obvious that all the corresponding coordinates $x_2, x_3, \ldots ., x_m$ would be absent from the independent m-point invariants; and, since the invariants of s points, $s > m$, are all made up of the m-point invariants, the former would be free from the variables

* See Lie's "Theorie der Transformationsgruppen," vol. III, Abtheilung V, or "Leipziger Berichte," vol. XLII.

$x_1, x_2, \ldots, x_s$. Among the differential equations defining these invariants, we should then necessarily find the following:

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \ldots, \quad \frac{\partial f}{\partial x_s} = 0; \tag{I}$$

s independent equations in all. But, the group considered being finite, containing ρ infinitesimal transformations, say, it could not give rise to more than ρ independent differential equations in any number of points. By choosing s sufficiently large, we see the impossibility of the system (I), and our assumption that one of the coordinates of the m points $x_1 y_1 \ldots, x_2 y_2 \ldots, \ldots, x_m y_m \ldots$ could be absent from the set of independent invariants in these m points is contradicted.

According to this law, it would, for example, be impossible so to choose the system of coordinates that the two-point invariant for the Euclidean Group of Motions in a plane, $(x_1 - x_2)^2 + (y_1 - y_2)^2$, would contain less than the four coordinates $x_1 y_1$, $x_2 y_2$, which is otherwise apparent.

From this law can be made the following important deduction:

*In all the groups that we are considering (defined above), no m infinitesimal transformations can have the same path-curves ("Bahncurven").**

In the following, Professor Lie's notation will be used. A group formed by ρ independent infinitesimal transformations, in three variables, say,

$$X_i f \equiv \xi_i(x, y, z)\frac{\partial f}{\partial x} + \eta_i(x, y, z)\frac{\partial f}{\partial y} + \zeta_i(x, y, z)\frac{\partial f}{\partial z}, \qquad i = 1, 2, \ldots, \rho,$$

we shall call the "ρ-parametric group $X_1 f \ldots X_\rho f$." It is convenient to write "R_n" for "space of n dimensions."

Lie has determined all groups in R_3 of the class defined above for which $m = 2$, as already mentioned. To these belong the Euclidean Group of Motions in R_3,

$$p, q, r, \quad yr - zq, \quad zp - xr, \quad xq - yp,$$

whose two-point invariant is the distance between the two points. (Here we have written p, q, r in place of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ respectively.)

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* A particular case of this law was given by Lie. See pp. 404–405 of his "Theorie der Transformationsgruppen," vol. III.

If we examine the Euclidean Group of Motions and Similar Transformations in R_3,

$$p,\ q,\ r, \quad yr - zq, \quad zp - xr, \quad xq - yp, \quad xp + yq + zr,$$

we find it to be a group of the class considered above, for which $m = 3$. Three points have two independent invariants, namely, two angles of the plane triangle with the three points as vertices. It might be of interest to find whether there are other groups of like character, i. e., *groups in three variables x, y, z, for which not less than three points have invariants, and for which the invariants of $s > 3$ points are all dependent upon the three-point invariants.*

Following the methods employed by Lie in his "Untersuchungen über die Grundlagen der Geometrie," and by paying attention to the laws given above for such groups, we find that they are *eight-*, *seven- or six-parametric* according as the number of independent three-point invariants is one, two or three.

The Eight-parametric Groups.

There are no eight-parametric primitive groups in three variables.* To such a group must, therefore, belong an invariant system of surfaces or curves.†

All groups in three variables, x, y, z, having a single invariant system of surfaces, but no invariant system of curves, and those having an invariant system of curves, say $x =$ const., $y =$ const., and no invariant system of surfaces of the form $\phi(x, y) =$ const., have been determined.‡ Only the following two types are eight-parametric, and have no three infinitesimal transformations with the same path-curves:

A
$$\boxed{\begin{array}{l} p,\ q,\ xq + r,\ xp - yq - 2zr,\ yp - z^2 r,\ xp + yq, \\ x^2 p + xyq + (y - xz)\,r,\ xyp + y^2 q + z\,(y - xz)\,r \end{array}}$$

B
$$\boxed{\begin{array}{c} p,\ q,\ xq,\ xp - yq,\ yp,\ xp + yq + r, \\ x^2 p + xyq + \tfrac{3}{2}xr,\ xyp + y^2 q + \tfrac{3}{2}\,yr \end{array}}$$

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* See chap. 7, vol. III, of "Theorie der Transformationsgruppen."

† We say that a group *has an invariant system of surfaces or curves*, or that such a system *belongs to the group*, when the members of that system are interchanged by the transformations of the group.

‡ See chap. 8, vol. III, of "Theorie der Transformationsgruppen."

The remaining eight-parametric groups must have an invariant system of curves, say $x = \text{const.}$, $y = \text{const.}$, and an invariant system of surfaces $\phi(x, y) = \text{const.}$, —say $x = \text{const.}$ They must, therefore, be of the form

$$X_i f \equiv \xi_i(x) p + \eta_i(x, y) q + \zeta_i(x, y, z) r, \qquad i = 1, 2, \ldots, 8.$$

The shortened groups ("verkürzte Gruppe"),

$$\overline{X}_i f \equiv \xi_i(x) p + \eta_i(x, y) q, \qquad i = 1, 2, \ldots$$

can be written down immediately from the tables of such groups.* It may be noticed that it follows from the laws given above that these shortened groups must be six-, seven- or eight-parametric, and that the shortened groups

$$\overline{\overline{X}}_i f \equiv \xi_i(x) p, \qquad i = 1, \ldots$$

must be three-parametric.

The remaining terms $\zeta_i(x, y, z) r$ of the infinitesimal transformations of these groups are then determined as Professor Lie has determined the corresponding terms in the groups considered in his "Untersuchungen über die Grundlagen der Geometrie." We would find the following groups:

Γ $\boxed{\begin{array}{c} p,\ q,\ xq + r,\ x^2 q + 2xr,\ x^3 q + 3x^2 r,\ x^4 q + 4x^3 r, \\ xp + 2yq + zr,\ x^2 p + 4xyq + (2xz + 4y) r \end{array}}$

Δ $\boxed{\begin{array}{c} p,\ q,\ xq + r,\ x^2 q + 2xr,\ x^3 q + 3x^2 r,\ yq + zr, \\ xp + yq,\ x^2 p + 3xyq + (xz + 3y) r \end{array}}$

E $\boxed{\begin{array}{c} r,\ p,\ q,\ xq,\ x^2 q + xr,\ yq + zr,\ xp + yq, \\ x^2 p + 2xyq + yr \end{array}}$

Z $\boxed{\begin{array}{c} r,\ p,\ q,\ xq,\ xr,\ 2xp + yq + zr,\ yq + (ay + bz) r, \\ x^2 p + xyq + xzr;\ b \neq 0,\ a(b - 1) = 0 \end{array}}$

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* See page 361 of Lie's "Continuerliche Gruppen."

The criterion that these groups possess the required properties resolves itself into the following conditions:

If

$$X_i^{(n)} f \equiv \xi_i (x_n, y_n, z_n) p_n + \eta_i (x_n, y_n, z_n) q_n + \zeta_i (x_n, y_n, z_n) r_n,$$
$$i = 1, 2, \ldots, 8$$

be any one of the groups written in the variables x_n, y_n, z_n, the eight partial differential equations

$$X_i^{(1)} f + X_i^{(2)} f + X_i^{(3)} f = 0, \qquad i = 1, 2, \ldots, 8$$

should be independent of each other.

If $I(a, b, c)$ represents the invariant in the points x_a, y_a, z_a; x_b, y_b, z_b; x_c, y_c, z_c, the four invariants

$$I(1, 2, 3), \quad I(1, 2, 4), \quad I(1, 3, 4), \quad I(2, 3, 4)$$

should be independent of each other.

The groups A and Γ, and Z, when $b = -1$, do not satisfy the last condition.*

Thus, *the eight-parametric groups satisfying the required conditions stated above are the groups* B, Δ *and* E, *and the group*

$$\mathrm{Z}' \quad \boxed{\begin{array}{c} r, p, q, \ xq, \ xr, \ 2xp + yq + zr, \ yq + (ay + bz) r, \\ x^2 p + xyq + xzr; \ b \neq 0 \text{ or } -1, \ a(b-1) = 0. \end{array}}$$

Here x, y, z may be regarded as complex variables. If we seek all typical groups with *real variables*, we would still have to find the real groups similar to the groups just given. The principles involved in the solution of this problem are clearly exhibited in Chapter 19 of Lie's "Theorie der Transformationsgruppen," vol. III. The real group-types are found to be

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*It is interesting to notice that in these cases the invariants $I(a, b, c)$ may be put in such form that we have identically

$$I(1, 2, 3) = I(1, 2, 4) + I(2, 3, 4) + I(3, 1, 4).$$

the groups B, Δ and E as given, Z′ with the additional condition that a and b are real constants, and the group

$$\mathrm{H}\quad \boxed{\begin{array}{c} r,\, p,\, q,\;\; xq,\;\; xr,\;\; 2xp + yq + zr,\;\; x^2 p + xyq + xzr, \\ yq + zr + c\,(zq - yr)\,;\; c \text{ real}, \end{array}}$$

which is similar to the group Z′ by means of an imaginary transformation of the variables.

The Seven-parametric Groups.

The seven-parametric groups are more numerous. Here three points have two independent invariants, which, together, must contain all the nine coordinates of the three points; and no three infinitesimal transformations of any one of the groups can have the same path-curves. By attending to these laws and proceeding as in the case of the eight-parametric groups, we find the seven-parametric groups without much difficulty.

The only primitive group is the before-mentioned Group of Euclidean Motions and Similar Transformations

$$\mathrm{A}\quad \boxed{p,\, q,\, r,\;\; yr - zq,\;\; zp - xr,\;\; xq - yp,\;\; xp + yq + zr}$$

From Chapter 8 of "Theorie der Transformationsgruppen," vol. III, we find the following groups:

$$\mathrm{B}\quad \boxed{p,\, r,\;\; q + xr,\;\; xq + \tfrac{1}{2}x^2 r,\;\; xp - yp,\;\; yp + \tfrac{1}{2}y^2 r,\;\; xp + yq + 2zr}$$

$$\Gamma\quad \boxed{p,\, q,\, r,\;\; xq,\;\; xp - yq,\;\; yp,\;\; zr + a\,(xp + yq)}$$

The remaining groups, about 25 in all, can be put in the form

$$X_i f \equiv \xi_i(x)\,p + \eta_i(x,\, y)\,q + \zeta_i(x,\, y,\, z)\,r, \qquad i = 1,\, 2,\, \ldots\ldots,\, 7,$$

where the shortened groups

$$\overline{X}_i f \equiv \xi_i(x)\,p + \eta_i(x,\, y)\,q, \qquad i = 1,\, 2,\, \ldots\ldots$$

are five-, six- or seven-parametric, and the shortened groups

$$\overline{\overline{X}}_i f \equiv \xi_i(x)\, p, \qquad i = 1, \ldots.$$

are two- or three-parametric.

The writer has not determined all the real groups belonging to this class, but has determined the *real primitive groups*, i. e., groups with no real invariant systems of surfaces or curves.* There are only two types of such groups, namely, the group A and one similar to this:

$$\mathrm{A}' \quad \boxed{p,\ q,\ r,\ yr + zq,\ xr + zp,\ xq - yp,\ xp + yq + zr}$$

The Six-parametric Groups.

The six-parametric groups under consideration are very numerous, being all the groups in three variables for which two points have no invariants and for which three points have three.† There are no primitive groups of this class, and, therefore, no real primitive.*

General Properties of these Groups.

There being no invariant relation connecting the coordinates of two points, any point in general position is free to move in any way in space if one point is fixed. All of the groups, excepting two, are imprimitive, and we should, therefore, expect that certain systems of points are restricted to move on certain loci if a given point is fixed.

In the case of the eight-parametric groups, if two points in general position are fixed, a third point, also in general position, must move on the surface

$$I(x_1, y_1, z_1;\ x_2, y_2, z_2;\ x, y, z) = I(x_1, y_1, z_1;\ x_2, y_2, z_2;\ x_3, y_3, z_3),$$

where $x_1, y_1, z_1;\ x_2, y_2, z_2$ are the two fixed points, x_3, y_3, z_3 the third point; x, y, z running coordinates, and $I(x_1$, etc.) the invariant in the three points.

* The writer has found that in R_3 all real primitive groups must be similar to those that are fully primitive, i. e., possess no invariant systems of curves or surfaces, real or imaginary.

† In fact, a ρ-parametric group in R_n will always belong to the wider class of groups defined above if ρ is a multiple of n, and there are no invariants for less than $\rho/n + 1$ points.

In the case of the seven-parametric groups, if two points in general position are fixed, any third point in general position must move on the curve

$$\begin{aligned} I(x_1, y_1, z_1;\ x_2, y_2, z_2;\ x, y, z) &= I(x_1, y_1, z_1;\ x_2, y_2, z_2;\ x_3, y_3, z_3), \\ J(x_1, y_1, z_1;\ \text{etc.} \qquad\qquad\quad) &= J(x_1, y_1, z_1;\ \text{etc.} \qquad\qquad\qquad), \end{aligned}$$

where I and J are the two three-point invariants, x_1, y_1, z_1, etc., as above. Thus, if two points are fixed in the case of the Euclidean Group of Motions and Similar Transformations, any third point must move on a circle whose plane is perpendicular to the straight line joining the two fixed points, and whose center lies in that line. This is also geometrically evident.

In the case of the six-parametric groups, if two points in general position are fixed, all other points are fixed.

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